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On structure-oriented hybrid two-stage iteration methods for the large and sparse blocked system of linear equations[☆]

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ABSTRACT

In this paper, we first present a class of structure-oriented hybrid two-stage iteration methods for solving the large and sparse blocked system of linear equations, as well as the saddle point problem as a special case. And the new methods converge to the solution under suitable restrictions, for instance, when the coefficient matrix is positive stable matrix generally. Numerical experiments for a model generalized saddle point problem are given, and the results show that our new methods are feasible and efficient, and converge faster than the *Classical Uzawa Method*.

Contents

1. Introduction

We consider the solution of systems of linear equations:

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 & V_1 \\ 0 & A_2 & \cdots & 0 & V_2 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & A_L & V_L \\ U_1 & U_2 & \cdots & U_L & A_{L+1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \\ x_{L+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_L \\ b_{L+1} \end{pmatrix}, \quad \text{or} \quad Ax = b, \tag{1.1}$$

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where $\hat{A} = \mathrm{diag}\begin{pmatrix} A_1 & A_2 & \cdots & A_L \end{pmatrix} \in R^{n\times n}$, denote the blocked diagonal matrix

$$C = -A_{L+1} \in R^{m\times m},$$
$$y = (x_1, x_2, \ldots, x_L)^T \in R^n, \qquad f = (b_1, b_2, \ldots, b_L)^T \in R^n, \qquad z = x_{L+1} \in R^m,$$

$g = b_{L+1} \in R^m$ and $m \le n$. We further assume that the matrix A is large and sparse; see [1–3].

The linear system (1.1) arises in a variety of scientific and engineering applications, including computational fluid dynamics, mixed finite element of elliptic *PDE*s, constrained optimization, constrained least-squares problem and so on. In large number of these applications, the linear system (1.1) is called a saddle and positive stable, when $r(U_i) = m$ for some i ($i \in [1, \ldots, L]$) namely full row-rank, and $C = 0$. In this case, the linear system (1.1) is called a classical saddle point problem, which was studied in many papers on iteration methods [1,2,4–18], such as Uzawa-type method [15–18], HSS iteration methods [19], preconditioned Krylov subspace iteration methods [20], restrictively preconditioned conjugate gradient methods [21]. However, there are other situations [22–29], among which the most notable is the numerical solution of the Navier–Stokes equations of fluid dynamics. When $\hat{A}$ is blocked diagonal matrix, $\hat{A} \ne \hat{A}^T$ and its symmetric part $H := \frac{1}{2}(\hat{A}+\hat{A}^T)$ is positive stable, $r(U) = m$, and C is symmetric and semi-stable, the linear system (1.1) is called generalized saddle point problem.

The linear system of Eq. (1.1) can be written in the following form

$$\begin{bmatrix} \hat{A} & U^T \\ U & -C \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{1.2}$$

Instead of solving (1.1), Bai and Golub [30] solved the following system equation:

$$\begin{bmatrix} \hat{A} & U^T \\ -U & C \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} \quad \text{or} \quad Ax = b. \tag{1.3}$$

The coefficient matrix of (1.3) has the following desirable properties.

Lemma 1.1 ([1, Theorem 2.2]). *Let $A \in R^{(n+m)\times(n+m)}$ be efficient matrix defined in (1.1). Assuming that $H := \frac{1}{2}(A + A^T)$ is positive stable, U has full row-rank, C is symmetrical and positive semi-stable, and $\ker(H) \cap \ker(U) = \emptyset$, where $\ker(\cdot)$ denotes the null-space of the corresponding matrix. Let $\sigma(A)$ denote the spectrum of A and $\lambda \in \sigma(A)$ is an eigenvalue of A. Then*

1. *A is non-singular;*
2. *A is semi-positive: $q \in R^{n+m}$;*
3. *A is positive stable: $\lambda > 0$ for all $\lambda \in \sigma(A)$.*

Thus by changing the sign of last m equation in (1.1), we can gain the positive stableness, then by appropriate transformation, the equivalent generalized saddle point system can be equivalently seen as the saddle point system.

Lemma 1.2 ([2, Theorem 3.1]). *Let $A, M \in R^{n\times n}$ be non-singular, $A = M - N$ is a splitting of the matrix A, and $T = M^{-1}N$. If*

$$x^T Ax \ne 0 \quad and \quad \frac{x^T Mx}{x^T Ax} > \frac{1}{2}, \tag{1.4}$$

*hold for any $x \in R^n$, $x \ne 0$, then we call $A = M - N$ a generalized P-*regular *splitting; if (1.4) holds for $x \in E_T$, $x \ne 0$, we call $A = M - N$ a local P-*regular *splitting; for the iteration matrix $T = M^{-1}N$, where we use E_T to denote the set of eigenvectors of the matrix T with, at least, one eigenvector being associated with each of its distinct eigenvalues. Obviously, when $A \in R^{n\times n}$ is symmetrical and non-singular, the splitting is a generalized P-*regular *splitting if and only if it is a P-*regular *splitting.*

Lemma 1.3 ([2, Definition 3.1]). *Let $A, M \in R^{(n+m)\times(n+m)}$ be non-singular and $A = M - N$. Denote $T = M^{-1}N$ and assume $x^T Mx \ne 0$ ($\forall x \in E_T, x \ne 0$). Then $\rho(T) < 1$ if and only if $A = M - N$ is a local P-*regular *splitting.*

In this paper, we will focus on these problems with certain structure like the large sparse system of linear equations (1.1), which provides a frame work for saddle point problem and successively rank-one updated problem. This kind of structure-oriented algorithm makes the most use of the sparseness of the coefficient matrix, which has attracted much attention of the research community (see [5,31]).

The remainder of the paper is arranged as follows. In Section 2, from the PHTI-method given in [1], we present a structure-oriented algorithm (*SHTI-method*) and its modified form (*MSHTI-method*). Then their convergence theories under some reasonable condition, namely, when the coefficient matrix is non-symmetrical positive stable matrix are established in Section 3. In Section 4, numerical experiments for VLSI circuit design problem are presented. The numerical results show that our new methods are feasible and efficient.

2. Iteration methods

In convenience of our subsequent statements, we stipulate that the considered multiprocessor system is made up of L CPUs. Let (R_i, S_i, E_i) $(i = 1, 2, \ldots, L)$ is a multisplitting of the matrix $A_{L+1} \in R^{n_{L+1}\times n_{L+1}}$, i.e. $R_i, S_i \in R^{n_{L+1}\times n_{L+1}}$ $(i = 1, 2, \ldots, L)$ are splitting matrices such that

$$A_{L+1} = R_i - S_i, \quad \det(R_i) \ne 0, \; i = 1, 2, \ldots, L, \tag{2.1}$$

in which R_i are nonnegative diagonal matrices and

$$\sum_{i=1}^{L} E_i = I, \quad I \in R^{n_{L+1}\times n_{L+1}} \text{ (Identity matrix)}, \tag{2.2}$$

for $i = 1, 2, \ldots, L$. Assuming (P_i, Q_i) is a splitting of the matrices $R_i \in R^{n_{L+1}\times n_{L+1}}$, i.e. $P_i, Q_i \in R^{n_{L+1}\times n_{L+1}}$ satisfy

$$R_i = P_i - Q_i, \quad \det(P_i) \neq 0, \tag{2.3}$$

(M_i, N_i) is a splitting of the matrix $A_i \in R^{n_{L+1}\times n_{L+1}}$, i.e. $M_i, N_i \in R^{n_{L+1}\times n_{L+1}}$ satisfy

$$A_i = M_i - N_i, \quad \det(M_i) \neq 0, \tag{2.4}$$

and (F_i, G_i) is a splitting of the matrix $M_i \in R^{n_{L+1}\times n_{L+1}}$, i.e. $F_i, G_i \in R^{n_i\times n_i}$ satisfy

$$M_i = F_i - G_i, \quad \det(F_i) \neq 0. \tag{2.5}$$

We will abbreviate the splittings satisfying Eqs. (2.1)–(2.3) as $(R_i : P_i, Q_i; S_i; E_i)$ $(i = 1, 2, \ldots, L)$ and call it a two-stage multisplitting of the matrix $A_{L+1} \in R^{n_{L+1}\times n_{L+1}}$. Analogously, for $i = 1, 2, \ldots, \alpha$ we will abbreviate the splittings satisfying Eqs. (2.4) and (2.5) as $(M_i : F_i, G_i; N_i;)$ $(i = 1, 2, \ldots, L)$ and call it a two-stage splitting of the matrix $A_i \in R^{n_{L+1}\times n_{L+1}}$.

Now, we consider the following iteration method called structure-oriented hybrid two-stage iteration method (*SHTI-method*) and the modified structure-oriented hybrid two-stage iteration method (*MSHTI-method*) for solving the large and sparse blocked linear equations (1.1) in the following.

Algorithm 2.0 (*SHTI-Method*).

$$y^{k,l} = E_l M_l^{-1}(N_l x^k + b),$$

$$x^{k+1} = \sum_{l=1}^{L} E_l y^{k,l}.$$

Algorithm 2.1 (*MSHTI-Method*).

$$x^{k+1} = \left(I - \sum_{l\in S_k}^{L} E_l\right) x^k + \sum_{l\in S_k}^{L} E_l M_l^{-1}(N_l x^k + b),$$

$$(z^{k,l})_j = (x^{k-d_{k,l,n}})_j.$$

Some basic restrictions on the parameters herein are:

j: $1 \le j \le L+1$.
S_k: subset of $\{1, \ldots, L+1\}$, nonempty, $0 \le k \le +\infty$.
E_l: nonnegative, nonzero, diagonal, $1 \le l \le L+1$, $\sum_l^{L+1} E_l$ non-singular.
$d_{k,l,n}$: $0 \le d_{k,l,n} \le sk$ for all $k \ge 0$, $l \in S_k$, $1 \le j \le L+1$.

If $A \in R^{(n+m)\times(n+m)}$ is a positive stable matrix, $b \in C^{n+m}$ and $x^{(0)} \in R^{n+m}$ are the initial guess, then this algorithm leads to the solution of system of linear equations (1.1). For $p = 0, 1, 2, \ldots$ until convergence,

1. Solve $x_i^{(p+1)} p = 0, 1, 2, \ldots$ through

$$x_i^{p,0} = x_i^{(p)},$$

$$\overline{x_{L+1}^{(p)}} = x_{L+1}^{(p)},$$

$$F_i x_i^{p,k+1} = G_i x_i^{p,k} + N_i x_i^{(p)} - V_i \overline{x_0^{(p)}} + b_i, \quad k = 0, 1, \ldots, k_0(i,p) - 1,$$

$$x_i^{(p+1)} = x_i^{p,k_i(i,p)},$$

$$\begin{cases} R_i x_{L+1}^{(p)} = S_i \overline{x_{L+1}^{(p)}} - U_i x_i^{(p+1)} - \displaystyle\sum_{l=i+1}^{L+1} U_i x_i^{(p)} + b_{L+1}, & \text{if } i = 1, \\ R_i x_{L+1}^{(p)} = S_i \overline{x_{L+1}^{(p)}} - \displaystyle\sum_{l=1}^{i-1} U_i x_i^{(p)} - U_i x_i^{(p+1)} - \displaystyle\sum_{l=i+1}^{L+1} U_i x_i^{(p)} + b_{L+1}, & \text{if } i = 2, \ldots, L. \end{cases}$$

2. Solve $x_{L+1}^{(p+1)} p = 0, 1, 2, \ldots,$ through

$$x_{L+1}^{p,i,0} = x_{L+1}^{(p)},$$

$$P_i x_{L+1}^{p,k+1} = Q_i x_{L+1}^{p,k} + S_i x_{L+1}^{(p)} - \sum_{l=1}^{L} U_l x_l^{(p+1)} + b_{L+1}, \quad k = 0, 1, \ldots, k_i(i,p) - 1,$$
$$x_{L+1}^{(p+1)} = x_{L+1}^{p,k_0(i,p)}, \quad i = 1, 2, \ldots, L.$$

3. Concurrently solve $x_{L+1}^{(p+1)}$ by

$$x_{L+1}^{(p+1)} = \sum_{i=1}^{L+1} E_i x_{L+1}^{p,i}$$

where $\{k_j(i,p)\}_{p=0}^{\infty}$, $i = 1, 2, \ldots, L$, are positive integer sequences which are chosen either at the beginning of the iteration or in the implementation of the method and

$$x^{(p)} = \left((x_0^{(p)})^T, (x_1^{(p)})^T, \ldots, (x_{L+1}^{(p)})^T\right)^T, \quad p = 0, 1, 2, \ldots.$$

On the other hand, the introduction of the modified step

$$R_i x_{L+1}^{(p)} = S_i \overline{x_{L+1}^{(p)}} - \sum_{l=1, l\neq i}^{L+1} U_i x_i^{(p)} - U_i x_i^{(p+1)} + b_{L+1}, \quad i = 1, \ldots, L;$$

makes Algorithm 2.1 to be implemented in such a manner that each processor of the multiprocessor system cannot carry out a varying number of local inner iterations until a mutual phase time is reached when all processors are ready to contribute to the global iteration. Then it avoids the asynchronous waiting among the processors. Hence, it greatly improves the parallel computing efficiency of Algorithm 2.0.

3. Convergence theories

After direct calculations we can rewrite Algorithm 2.0 in the following concise forms:

$$\begin{cases} x_i^{(p+1)} = \left[(F_i^{-1}G_i)^{k_i(i,p)} + \sum_{k=0}^{k_i(i,p),} - \sum_{k=0}^{k_i(i,p)-1} (F_i^{-1}G_i)^k F_i^{-1} V x_{L+1}^{(p)} + \sum_{k=0}^{k_i(i,p)-1} (F_i^{-1}G_i)^k F_i^{-1} b_i, (p = 0, 1, 2, \ldots)\right] \\ x_{L+1}^{(p+1)} = \sum_{i=1}^{L} E_i \left[(P_i^{-1}Q_i)^{k_i(i,p)} + \sum_{k=0}^{k_0(i,p)-1} (P_i^{-1}Q_i)^k P_i^{-1} S_i\right] x_{L+1}^{p} - \sum_{i=1}^{L} E_i \sum_{k=0}^{k_0(i,p)-1} (P_i^{-1}Q_i)^k P_i^{-1} \sum_{i=1}^{L} U_i x_i^{(p+1)} \\ \quad + \sum_{i=1}^{L} E_i \sum_{k=0}^{k_0(i,p)-1} (P_i^{-1}Q_i)^k P_i^{-1} b_{L+1}, \quad (p = 0, 1, 2, \ldots). \end{cases} \tag{3.1}$$

Let us introduce matrices and vectors

$$\begin{cases} H_{L+1,p} = \sum_{i=1}^{L} E_i \left[\left(P_i^{-1}Q_i\right)^{k_i(i,p)} + \sum_{k=0}^{k_0(i,p)-1} \left(P_i^{-1}Q_i\right)^k P_i^{-1} S_i\right], \\ G_{L+1,p} = \sum_{i=1}^{L+1} E_i \sum_{k=0}^{k_0(i,p)-1} \left(P_i^{-1}Q_i\right)^k P_i^{-1}, \qquad b_{L+1,p} = G_{L+1,p} b_{L+1}, \\ H_{i,p} = (F_i^{-1}G_i)^{k_i(i,p)} + \sum_{k=0}^{k_i(i,p)-1} (F_i^{-1}G_i)^k F_i^{-1} N_i, \\ G_{i,p} = \sum_{k=0}^{k_i(i,p)-1} (F_i^{-1}G_i)^k F_i^{-1}, \qquad b_{i,p} = G_{i,p} b_{L+1}, \\ V_{i,p} = -G_{i,p} V_i, \qquad U_{i,p} = -G_{L+1,p} U_i, \end{cases} \tag{3.2}$$

for $i = 1, 2, \ldots, L$. Then the above relations can be equivalently expressed as

$$\begin{cases} x_i^{(p+1)} = H_{i,p} x_i^{(p)} + V_{i,p} x_{L+1}^{(p)} + b_{i,p}, \quad i = 1, 2, \ldots, L \\ x_{L+1}^{(p+1)} = H_{L+1,p} x_{L+1}^{(p)} + \sum_{l=1}^{L} U_{l,p} x_l^{(p+1)} + b_{L+1,p}. \end{cases} \tag{3.3}$$

Furthermore, if we define error vector

$$\varepsilon^{(p)} = \left(\left(\varepsilon_1^{(p)}\right)^T, \left(\varepsilon_2^{(p)}\right)^T, \ldots, \left(\varepsilon_{L+1}^{(p)}\right)^T\right)^T = x^{(p)} - x^*,$$

where x^* is the unique solution of the large and sparse blocked system of linear equations (1.1). From Eq. (3.3) we can obtain that

$$\varepsilon^{(p+1)} = T_{\mathrm{SHTI}}^{(p)}\varepsilon^{(p)}, \quad p = 0, 1, 2, \ldots \tag{3.4}$$

with

$$T_{\mathrm{SHTI}}^{(p)} = \begin{pmatrix} H_{1,p} & 0 & \cdots & 0 & V_{1,p} \\ 0 & H_{2,p} & \cdots & 0 & V_{2,p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & H_{L,p} & V_{L,p} \\ U_{1,p}H_{1,p} & U_{2,p}H_{2,p} & \cdots & U_{L,p}H_{L,p} & H_{L+1,p} + \sum_{i=1}^{L} U_{i,p}V_{i,p} \end{pmatrix}. \tag{3.5}$$

Therefore, to prove the convergence of method 2.0 we only need to verify that the error sequence determined by Eqs. (3.4) and (3.5) satisfy $\varepsilon^{(p)} \to 0$ for $p \to \infty$. In the following, we will mainly discuss the sufficient conditions that can guarantee the validity of this fact. Firstly, we consider the situation when the coefficient matrix $A \in C^{n\times n}$ is non-symmetrical positive stable and non-singular. From Eq. (3.2), we can obtain that

$$\begin{cases} I - H_{i,p} = \left(I - (F_i^{-1}G_i)^{k_i(i,p)}\right)\left(I - (I - F_i^{-1}G_i)^{-1}F_i^{-1}N_i\right) \\ \quad = \left(I - (F_i^{-1}G_i)^{k_i(i,p)}\right)(I - F_i^{-1}G_i)^{-1}\left(I - F_i^{-1}G_i - F_i^{-1}N_i\right) \\ \quad = \left(I - (F_i^{-1}G_i)^{k_i(i,p)}\right)(I - F_i^{-1}G_i)^{-1}F_i^{-1}A_i, \end{cases}$$

for $i = 1, 2, \ldots, L;\ p = 0, 1, 2, \ldots$, and A_i are non-singular. It is obvious that $I - H_{i,p}$ is non-singular if and only if 1 is not an eigenvalue of either $F_i^{-1}G_i$ or $(F_i^{-1}G_i)^{k_j(i,p)}$. Therefore, we learn that there exists a unique pair of matrices $M_{H_{i,p}}$ and $N_{H_{i,p}}$ such that

$$A_i = M_{H_{i,p}} - N_{H_{i,p}} \quad \text{and} \quad H_{i,p} = M_{H_{i,p}}^{-1}N_{H_{i,p}}.$$

Moreover, matrices $M_{H_{i,p}}$ and $N_{H_{i,p}}$ are defined as

$$M_{H_{i,p}} = F(I - F_i^{-1}G_i)(I - (F_i^{-1}G_i)^{k_j(i,p)})^{-1} \quad \text{and} \quad N_{H_{i,p}} = M_{H_{i,p}} - A_i. \tag{3.6}$$

Based on the above preparations, we can establish the following theorem for methods 2.0 and 2.1 when the coefficient matrix $A \in R^{(n+m)\times(n+m)}$ of the linear system (1.1) is positive stable.

Theorem 3.1. *Let $A \in R^{n\times n}$ be a positive stable matrix, $(R_i : P_i, Q_i; S_i; E_i)$ $(i = 1, 2, \ldots, L)$ is a two-stage multisplitting of the matrix $A_{L+1} \in R^{n_{L+1}\times n_{L+1}}$ such that each of the splittings $A_{L+1} = R_i - S_i$, $(i = 1, 2, \ldots, L)$ is a generalized P-regular splitting and each of the splittings $R_i = P_i - Q_i$, $(i = 1, 2, \ldots, L)$ is a local P-regular splitting, and $(M_i : F_i, G_i; N_i;)$ is a two-stage splitting of the matrix $A_i = M_i - N_i$ $(i = 1, 2, \ldots, L)$ is generalized P-regular splittings and $M_i = F_i - G_i$ $(i = 1, 2, \ldots, L)$ is a symmetrical and convergent splittings of $M_i \in R^{n_i\times n_i}$, which is a symmetrical positive stable matrix. Then for any initial vector $x^{(0)} \in R^n$ Algorithm 2.0 converges to the unique solution of the large and sparse blocked system of linear equations (1.1) independent of the positive integer sequences*

$$\{k_j(i,p)\}_{p=0}^{\infty}, \quad i = 1, 2, \ldots, L;\ j = 1, \ldots, L+1;$$

provided

$$k_j(i,p) \geq 1, \quad i = 1, 2, \ldots, L;\ j = 1, \ldots, L+1;\ p = 0, 1, 2, \ldots.$$

Proof. Since $A \in R^{n\times n}$ is a positive stable matrix, we see that $A_i \in R^{n_i\times n_i}$ $(i = 0, 1, 2, \ldots, L)$ are all positive stable matrices. In accordance with Lemma 1.1 we know that there exists a positive vector $u_i \in R^{n_i}$, $i = 1, \ldots, L$, such that

$$u_i := A_iu_i > 0, \quad i = 1, \ldots, L.$$

On the other hand, from Eqs. (2.2)–(2.4) we can obtain that

$$\begin{cases} H_{L+1,p} = \sum_{i=1}^{L} E_i\left[(P_i^{-1}Q_i)^{k_i(i,p)} + \sum_{k=0}^{k_0(i,p)-1}(P_i^{-1}Q_i)^kP_i^{-1}(R_i - A_{L+1})\right] \\ \quad = (P_i^{-1}Q_i)^{k_i(i,p)} + \sum_{i=1}^{\alpha} E_i\left[\sum_{k=0}^{k_0(i,p)-1}(P_i^{-1}Q_i)^kP_i^{-1}(P_i - Q_i) + \sum_{k=0}^{k_0(i,p)-1}(P_i^{-1}Q_i)^kP_i^{-1}A_{L+1}\right] \\ \quad = I - \sum_{i=1}^{L} E_i \sum_{k=0}^{k_0(i,p)-1}(P_i^{-1}Q_i)^kP_i^{-1}A_{L+1} = I - G_{L+1,p}A_{L+1}. \end{cases}$$

And from Eqs. (3.5) and (3.6) we can analogously learn that $H_{i,p} = I - G_{i,p}A_i$, $i = 1, 2, \ldots, L$. Obviously, under the conditions of the theorem the matrices $H_{i,p}$ and $G_{i,p}$ are nonnegative matrices for all $i = 1, 2, \ldots, L$ and $p = 0, 1, 2, \ldots$. Hence, based on Eqs. (3.5) and (3.6) we have

$$H_{L+1,p}u_{L+1} = u_{L+1} - G_{L+1,p}u_{L+1} \le u_{L+1} - \sum_{i=1}^{L} E_i P^{-1} v_{L+1} < u_{L+1}.$$

Meanwhile, $M_i = F_i - G_i$ $(i = 1, 2, \ldots, \alpha)$ is a symmetrical and convergent splitting, it holds that $\rho(F_i^{-1}G_i) < 1$ $(i = 1, 2, \ldots, L)$. Hence,

$$\rho((F_i^{-1}G_i)^{k_i(i,p)}) = \rho(F_i^{-1}G_i)^{k_i(i,p)} < 1.$$

As

$$\begin{cases} M_{H_{i,p}} = F(I - F_i^{-1}G_i)(I - (F_i^{-1}G_i)^{k_j(i,p)})^{-1} \\ \quad = M_i(I - (F_i^{-1}G_i)^{k_j(i,p)})^{-1} \\ \quad = M_i \displaystyle\sum_{p=0}^{+\infty} (F_i^{-1}G_i)^{k_j(i,p)} \end{cases}$$

for $i = 1, 2, \ldots, L;\ j = 1, \ldots, L;\ p = 0, 1, 2, \ldots$. By applying (3.6) we have

$$\begin{cases} M_{H_{i,p}} + N_{H_{i,p}} = 2M_i(I - (F_i^{-1}G_i)^{k_j(i,p)})^{-1} - A \\ \quad = 2M_i((I - (F_i^{-1}G_i)^{k_j(i,p)})^{-1} - I) + M_i + N_i \\ \quad = 2M_i \displaystyle\sum_{p=1}^{+\infty} (F_i^{-1}G_i)^{k_j(i,p)} + M_i + N_i. \end{cases} \tag{3.7}$$

In addition, for any even positive integer p, we have

$$\begin{cases} M_i(F_i^{-1}G_i)^{k_j(i,p)} = F(I - F_i^{-1}G_i)(F_i^{-1}G_i)^{k_j(i,p)} \\ \quad = F(F_i^{-1}G_i)^{\frac{k_j(i,p)}{2}}(I - F_i^{-1}G_i)(F_i^{-1}G_i)^{\frac{k_j(i,p)}{2}} \\ \quad = \left(G_iF_i^{-1}\right)^{\frac{k_j(i,p)}{2}} M_i(F_i^{-1}G_i)^{\frac{k_j(i,p)}{2}}. \end{cases} \tag{3.8}$$

Because $A_i = M_i - N_i$ is asymmetrical splitting, we know that M_i, F_i and G_i are symmetrical matrices. Moreover, M_i is symmetrical positive stable. Eq. (3.8) shows that $M_i(F_i^{-1}G_i)^{k_j(i,p)}$ are symmetrical positive semi-stable matrices for all positive integers p. Therefore, $M_{H_{i,p}}$ is a symmetrical positive stable matrix. Eq. (3.7) shows that the symmetrical parts of $2M_{H_{i,p}} - A_i$ are exactly the symmetrical parts of $2M_i - A_i$ plus the symmetrical positive semi-stable matrices $2\sum_{p=1}^{+\infty} M_i(F_i^{-1}G_i)^{k_j(i,p)}$, i.e.

$$K(2M_{H_{i,p}} - A_i) = K(2M_i - A_i) + 2\sum_{p=1}^{+\infty} M_i(F_i^{-1}G_i)^{k_j(i,p)}$$

and the non-symmetrical part of $2M_{H_{i,p}} - A_i$ are exactly the symmetrical parts of $2M_i - A_i$, i.e. $K(2M_{H_{i,p}} - A_i) = K(2M_i - A_i)$. Because $A \in C^{n\times n}$ is a positive stable matrix and $A_i = M_i - N_i$ is a generalized P-regular splitting, we learn that $A = M_{H_{i,p}} - N_{H_{i,p}}$ is a generalized P-regular splitting, too. Now, by Lemma 1.3 we immediately know that

$$\rho\left(H_{i,p}\right) = \rho\left(M_{H_{i,p}}^{-1}N_{H_{i,p}}\right) < 1, \quad i = 1, 2, \ldots, L;\ p = 0, 1, 2, \ldots.$$

Likewise, based on Eqs. (3.5) and (3.6) we have

$$H_{i,p}u_i < u_i, \quad i = 1, 2, \ldots, L;\ p = 0, 1, 2, \ldots.$$

So there exist constants $\bar{\theta}_i \in [0, 1)$, $(i = 1, 2, \ldots, L)$ such that

$$H_{i,p}u_i < \theta_i u_i, \quad i = 1, 2, \ldots, L;\ p = 0, 1, 2, \ldots.$$

Hence the matrices $G_{i,p}$ for $i = 1, 2, \ldots, L$, $p = 0, 1, 2, \ldots$ are non-singular. Introduce matrices

$$M_p = \begin{pmatrix} G_{1,p}^{-1} & 0 & \cdots & 0 & 0 \\ 0 & G_{2,p}^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & G_{L,p}^{-1} & 0 \\ U_1 & U_2 & \cdots & U_L & G_{L+1,p}^{-1} \end{pmatrix}, \quad p = 0, 1, 2, \ldots,$$

$$N_p = \begin{pmatrix} G_{1,p}^{-1}H_{1,p} & 0 & \cdots & 0 & -V_1 \\ 0 & G_{2,p}^{-1}H_{2,p} & \cdots & 0 & -V_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \cdots & G_{L,p}^{-1}H_{L,p} & -V_L \\ 0 & 0 & \cdots & 0 & G_{L+1,p}^{-1}H_{L+1,p} \end{pmatrix}, \quad p = 0, 1, 2, \ldots.$$

By direct manipulations we know that holding

$$A = M_p - N_p, \qquad T_{\text{SHTI}}^{(p)} = M_p^{-1}N_p, \quad p = 0, 1, 2, \ldots, \tag{3.9}$$

and

$$M_p^{-1} = \begin{pmatrix} G_{1,p} & 0 & \cdots & 0 & 0 \\ 0 & G_{2,p} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & G_{L,p} & 0 \\ -G_{1,p}U_1G_{L+1,p} & -G_{1,p}U_2G_{L+1,p} & \cdots & -G_{1,p}U_LG_{L+1,p} & G_{L+1,p} \end{pmatrix}.$$

We can easily deduce from Eq. (3.9) that

$$M_p^{-1}\bar{v} \geq \begin{pmatrix} 1-\theta_1 & & & \\ & 1-\theta_2 & & \\ & & \ddots & \\ & & & 1-\theta_{L+1} \end{pmatrix} \bar{v} \geq \left(1-\bar{\theta}\right)\bar{v}, \quad p = 0, 1, 2, \ldots,$$

where $\bar{\theta} = \max_{0 \leq j \leq L} \bar{\theta}_j \in [0, 1)$. Now, from Eq. (3.9) we have

$$T_{\text{SHTI}}^{(p)}\bar{u} = M_p^{-1}N_p\bar{u} = \left(I - M_p^{-1}A\right)\bar{u} = \bar{u} - M_p^{-1}\bar{v} \leq \bar{u} - \left(1-\bar{\theta}\right)\bar{v} < \bar{u}.$$

Hence, there exists a constant $\theta \in [0, 1)$ such that

$$T_{\text{SHTI}}^{(p)}\bar{u} \leq \theta\bar{u}, \quad p = 0, 1, 2, \ldots.$$

Note from Eq. (3.5) that

$$T_{\text{SHTI}}^{(p)} \geq 0 \quad (p = 0, 1, 2, \ldots).$$

Let $\delta > 0$, such that the initial vector $\varepsilon^{(0)}$ corresponding to method 2.0 satisfies

$$\begin{aligned} \left|\varepsilon^{(p+1)}\right| = \left|T_{\text{SHTI}}^{(p)}\varepsilon^{(p)}\right| &\leq \left|T_{\text{SHTI}}^{(p)}\right|\left|T_{\text{SHTI}}^{(p-1)}\right|\cdots\left|T_{\text{SHTI}}^{(0)}\right|\left|\varepsilon^{(0)}\right| \\ &\leq \left|T_{\text{SHTI}}^{(p)}\right|\left|T_{\text{SHTI}}^{(p-1)}\right|\cdots\left|T_{\text{SHTI}}^{(0)}\right|(\delta\bar{u}) \leq \delta\theta^{p+1}\bar{u} \to 0 \quad (p \to \infty). \end{aligned}$$

This immediately implies $\varepsilon^{(p)} \to 0\,(p \to \infty)$. □

Theorem 3.2. *Let $A \in R^{n\times n}$ be a positive stable matrix, $(R_i : P_i, Q_i; S_i; E_i)$ $(i = 1, 2, \ldots, L)$ is a two-stage multisplitting of the matrix $A_{L+1} \in R^{n_{L+1}\times n_{L+1}}$ such that each of the splittings $A_{L+1} = R_i - S_i$, $(i = 1, 2, \ldots, L)$ is a generalized P-regular splitting and each of the splittings $R_i = P_i - Q_i$, $(i = 1, 2, \ldots, L)$ is a local P-regular splitting, and $(M_i : F_i, G_i; N_i;)$ is a two-stage splitting of the matrix $A_i = M_i - N_i$ $(i = 1, 2, \ldots, L)$ is generalized P-regular splittings and $M_i = F_i - G_i$ $(i = 1, 2, \ldots, L)$ is a symmetrical and convergent splitting of $M_i \in R^{n_i\times n_i}$ which is a symmetrical positive stable matrix. Then for any initial vector $x^{(0)} \in R^n$ Algorithm 2.1 converges to the unique solution of the large and sparse blocked system of linear equation (1.2) independently of the positive integer sequences*

$$\{k_j(i, p)\}_{p=0}^{\infty}, \quad i = 1, 2, \ldots, L;\ j = 1, \ldots, L+1$$

provided

$$k_j(i, p) \geq 1, \quad i = 1, 2, \ldots, L;\ j = 1, \ldots, L+1;\ p = 0, 1, 2, \ldots.$$

Proof. In the same condition as Theorem 3.1 we have

$$\bar{H}_{i,p} = I - \bar{G}_{i,p}A_i, \quad i = 1, 2, \ldots, L.$$

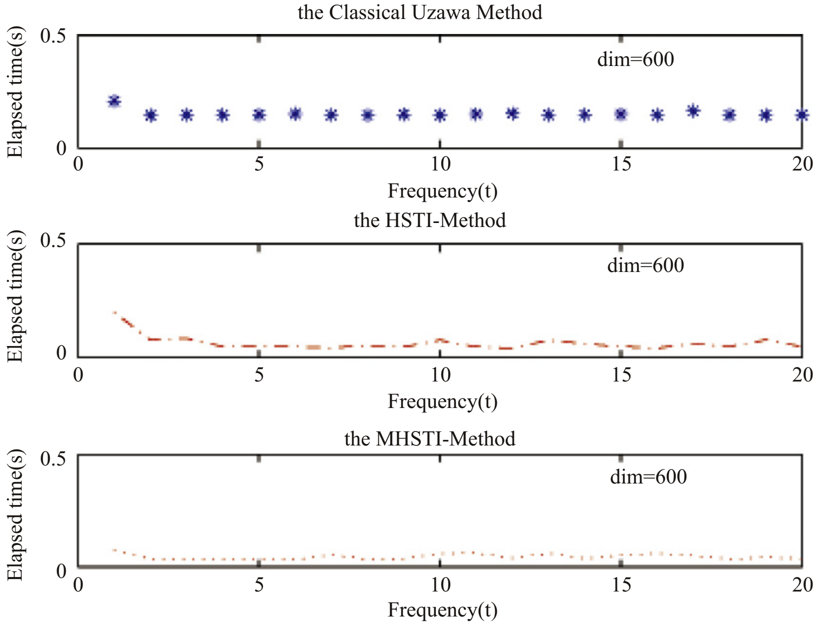

Fig. 1. dim = 600.

We only need to prove that $\forall j \in S_k$: subset of $\{1, \dots, L+1\}$, nonempty, $0 \le k \le +\infty$, the Algorithm 2.1 converges. Assuming that $x_j^{(p+1)} = H_{j,p}x_j^{(p)} + V_{j,p}x_{L+1}^{(p)} + b_{j,p}$, then

$$x_{j+1}^{(p+1)} = H_{j+1,p}x_{j+1}^{(p)} + V_{j+1,p}x_{L+1}^{(p)} + b_{j+1,p},$$

$$\rho\left(H_{i,p}\right) = \rho\left(M_{H_{i,p}}^{-1}N_{H_{i,p}}\right) < 1, \quad i = 1, 2, \dots, L;\ p = 0, 1, 2, \dots.$$

And from the process of Algorithm 2.1, we see that $|H_{0,p}x_j^{(p+1)}| < |x_j^{(p+1)}|$. Because $\rho(H_{0,p}) < 1$, $\bar{U}_{j,p} = -G_{L+1,p}U_j \le U_{j,p}$, $\bar{H}_{j,p} \le H_{j,p}$.

By Theorem 3.1, $\rho(\bar{H}_{j,p}) \le \rho(H_{j,p})$.

Similarly, we have for any initial vector $\varepsilon^{(0)}$, this immediately implies $\varepsilon^{(p)} \to 0\,(p \to \infty)$, which implies that the Algorithm 2.1 converges to the unique solution of linear equation. $\square$

4. Numerical experiments

In this section we will present some numerical experiments to compare our new methods with the classical Uzawa method and *PHTI*-method for VLSI circuit design problem. We apply this as an example to solve imitatively the large and sparse blocked linear equations (1.1) for which $\alpha = 2$, $n_i = \bar{n}$ $(i = 0, 1, 2)$ hence $n_i = 3\bar{n}$, $A_0 = A_i = \bar{A} \in R^{\bar{n}\times\bar{n}}$, $U_i = V_i = -I \in R^{\bar{n}\times\bar{n}}$, $i = 1, 2$ and

$$\bar{A} = \begin{pmatrix} \bar{B} & & -I \\ & \bar{B} & -I \\ -I & -I & \bar{B} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in R^n$$

where

$$\bar{B} = \begin{pmatrix} 8 & -2 & & & \\ -2 & 8 & -2 & & \\ & \ddots & \ddots & \ddots & \\ & & -2 & 8 & -2 \\ & & & -2 & 8 \end{pmatrix} \in R^{\bar{n}\times\bar{n}}.$$

The inner iteration numbers, namely, the positive integer sequences

$$\left\{k_j(i, p)\right\}_{p=0}^{\infty}, \quad i = 1, 2; j = 1, 2, 3$$

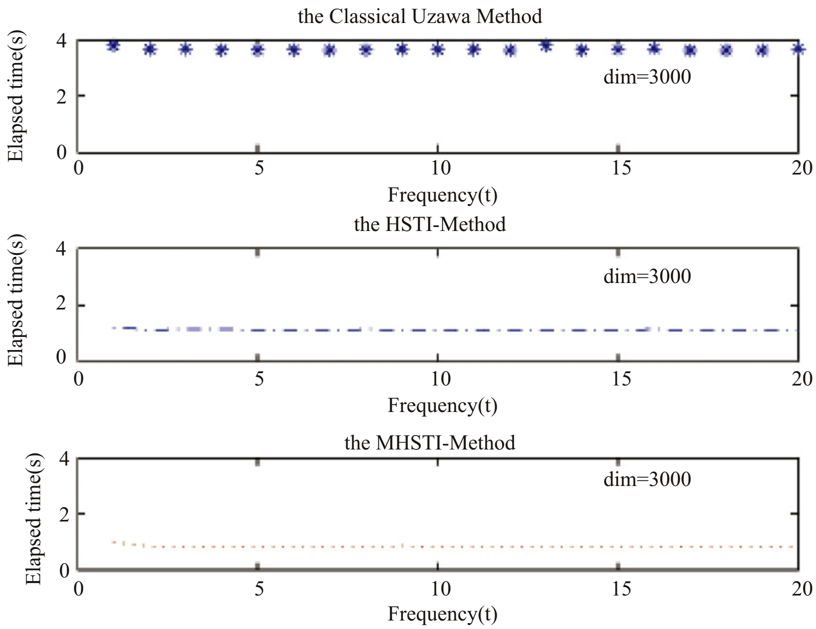

Fig. 2. dim = 3000.

are fixed to be $\{K\}_{p=0}^{\infty}$. All our computations are started with an initial vector possessing all components equal to zero and terminated once the current iterations $x^{(p)}$ obey $\|b - Ax^{(p)}\| \le 0.000001$. The dimension of the system of linear equations fixed 600 and 3000. We denote the Dimension of the coefficient matrix A of linear equations as dim. The elapsed time is listed in Figs. 1 and 2 to show the numerical behavior of our new method. We learn that the elapsed time only depends on the dimension n of the problem. All tests are started from zero vectors, performed in MATLAB with machine precision 10^{-6}.

Memory: 1 G; CPU: T5750; HD: 250 G; time: second.

From the graphs (Figs. 1 and 2), although the dimension is 600 or 3000, we can see that the *SHTI-method* is faster than the *Classical Uzawa method* in the test, and under the same condition the *MSHTI-method* is better than the *SHTI-method*. The numerical results show that the feasibility and efficiency of the *MSHTI-method* is the best of all above.

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